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Quantum evolution across singularities

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Abstract: Attempts to consider evolution across space-time singularities often lead to quantum systems with time-dependent Hamiltonians developing an isolated singularity as a function of time. Examples include matrix theory in certain singular time-dependent backgounds and free quantum fields on the two-dimensional compactified Milne universe. Due to the presence of the singularities in the time dependence, the conventional quantum-mechanical evolution is not well-defined for such systems. We propose a natural way, mathematically analogous to renormalization in conventional quantum field theory, to construct unitary quantum evolution across the singularity. We carry out this procedure explicitly for free fields on the compactified Milne universe and compare our results with the matching conditions considered in earlier work (which were based on the covering Minkowski space).

Keywords: M(atrix) Theories, Spacetime Singularities.

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1. Introduction

Dynamical evolution across space-time singularities is one of the most tantalizing, even if speculative, questions in modern theoretical physics. Should our theories point towards a beginning of time, it is very natural to ask what came before, and, indeed, whether there could be anything before.

In certain model contexts, quantum evolution across space-time singularities appears to be described by time-dependent Hamiltonians developing an isolated singularity as a function of time at the moment the system reaches a space-time singularity. It is then worthwhile to study such quantum Hamiltonians and establish some general prescriptions for using them to constuct a unitary quantum evolution. Needless to say, additional specifications are needed in a Schrödinger equation involving this kind of Hamiltonians, on account of the singular time dependence.

One of the simplest examples of such singular time-dependent Hamiltonians in systems with space-time singularities is given by a free scalar field on the Milne orbifold (see [1 – 5] and references therein for some recent occurrences of the Milne orbifold in models of cosmological singularities). We shall give a detailed consideration of this case in section 3. Here, it should suffice to say that the square root determinant of the metric of the Milne orbifold vanishes as $|t|$ when t goes to 0. Because of that, the kinetic term in the Lagrangian for a free field ϕ on the Milne orbifold will have the form $|t|(\partial_t\phi)^2$, and the corresponding term in the Hamiltonian expressed through the canonical momentum π_ϕ conjugate to ϕ will have the form $\pi_\phi^2/|t|$, which manifestly displays an $1/|t|$ singularity. The position of this singularity in the time dependence coincides with the metric singularity of the Milne orbifold.

While it is well-known that free fields on the Milne orbifold are not a good approximation to interacting systems, especially in gravitational theories [6, 7], analogous singular

time dependences have recently appeared in other models, which have been the main motivation for the present work. For example, 11-dimensional quantum gravity with one compact dimension in a certain singular time-dependent background with a light-like isometry is conjectured to be described by a time-dependent modification of matrix string theory [8, 9]. This model can be recast in the form of a (1+1)-dimensional super-Yang-Mills theory on the Milne orbifold. It will thus contain in its Hamiltonian the $1/|t|$ time dependence typical of the general Milne orbifold kinematics. The question of transition through the singularity will then amount to defining a quantum system with such singular Hamiltonian. Likewise, for the time-dependent matrix models of [10], which are conjectured to describe quantum gravity in non-compact eleven-dimensional time-dependent background with a light-like singularity, one obtains a quantum Hamiltonian with a singular time dependence.

In view of these examples, our present paper will address the question of how one should define unitary quantum evolution in the presence of isolated singularities in the time dependence of quantum Hamiltonians. Upon giving a general prescription for treating such singularities and discussing the ambiguities it incurs, we shall proceed with analyzing the simple yet instructive case of a free scalar field on the Milne orbifold. We shall further discuss the relation between our prescription and the recipes for quantum evolution of this system previously proposed in the literature (and based on considerations in the covering Minkowski space) [11 – 13, 7].

2. Isolated singularities in time-dependent quantum Hamiltonians

Following the general remarks in the introduction, we shall consider a quantum system described by the following time-dependent Hamiltonian:

$$H(t) = f(t,\varepsilon)h + H_{\rm reg}(t), \tag{2.1}$$

where $H_{\rm reg}(t)$ is non-singular around $t = 0$, whereas the numerical function $f(t,\varepsilon)$ develops an isolated singularity at $t = 0$ when ε goes to 0 (ε serves as a singularity regularization parameter), and h is a time-independent operator. We shall be interested in the evolution operator from small negative to small positive time. In this region, we shall assume that we can neglect the regular part of the Hamiltonian $H_{\rm reg}(t)$ compared to the singular part.[1] The Schrödinger equation takes the form

$$i\frac{d}{dt}|\Psi\rangle = f(t,\varepsilon)h|\Psi\rangle. \tag{2.2}$$

The solution for the corresponding evolution operator is obviously given by

$$\mathrm{U}(t,t') = \exp\left[-i\int_t^{t'} dt f(t,\varepsilon)h\right]. \tag{2.3}$$

[1] This assumption is actually stronger than one might naïvely have thought: seemingly small interaction terms in the Hamiltonian are sometimes responsible for large quantum effects, for instance due to degrees of freedom becoming light. An example where this happens is the matrix big bang model [8, 9], where an important one-loop potential is generated in the weak coupling region of the field theory. For this reason, our present discussion will not directly apply to the matrix big bang model, though we hope to treat that model using similar techniques in future work.

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When the regularization parameter ε is sent to 0, $f(t,\varepsilon)$ becomes singular and $\mathrm{U}(t,t')$ is in general not well-defined.

The goal is then to modify the Hamiltonian locally at $t=0$ in such a way that the evolution away from $t=0$ remains as it was before, but there is a unitary transition through $t=0$. Of course, a large amount of ambiguity is associated with such a program, and we shall comment on it below.

The most conservative approach to the Hamiltonian modification is suggested by (2.3). Since the problem arises due to the impossibility of integrating $f(t,\varepsilon)$ over t at $\varepsilon=0$, the natural solution is to modify $f(t,\varepsilon)$ locally around (in the ε-neighborhood of) $t=0$ in such a way that the integral can be taken (note that we are leaving the operator structure of the Hamiltonian intact).

The subtractions necessary to appropriately modify $f(t,\varepsilon)$ are familiar from the theory of distributions. Namely, for any function $f(t,\varepsilon)$ developing a singularity not stronger than $1/t^p$ as ε is sent to 0, with an appropriate choice of $c_n(\varepsilon)$, one can introduce a modified

$$\tilde{f}(t,\varepsilon) = f(t,\varepsilon) - \sum_{n=0}^{p-1} c_n(\varepsilon)\delta^{(p)}(t) \tag{2.4}$$

(where $\delta^{(p)}(t)$ are derivatives of the δ-function) in such a way that the $\varepsilon\to 0$ limit of $\tilde{f}(t,\varepsilon)$ is defined in the sense of distributions. The latter assertion would imply that the $\varepsilon\to 0$ limit of

$$\int \tilde{f}(t,\varepsilon)\mathcal{F}(t)dt \tag{2.5}$$

is defined for any smooth "test-function" $\mathcal{F}(t)$, and, in particular, that the $\varepsilon\to 0$ limit of (2.3) becomes well-defined, if $f(t,\varepsilon)$ is replaced by $\tilde{f}(t,\varepsilon)$. (Note that, since $f(t,\varepsilon)$ and $\tilde{f}(t,\varepsilon)$ only differ in an infinitesimal neighborhood of $t=0$, this modification will not affect the evolution at finite t).

As a matter of fact, the subtraction needed for our particular case is simpler than (2.4). Since the $n>0$ terms in (2.4) can only affect the value of the evolution operator (2.3) at $t'=0$, if one is only interested in the values of the wave function for non-zero times, one can simply omit the $n>0$ terms from (2.4). One can then write down the subtraction explicitly as

$$\tilde{f}(t,\varepsilon) = f(t,\varepsilon) - \left(\int_{-t_0}^{t_0} f(t,\varepsilon)dt\right)\delta(t). \tag{2.6}$$

The appearance of a free numerical parameter (which can be chosen as t_0 in the expression above, or a function thereof) is not surprising, since, if $\tilde{f}(t,\varepsilon)$ is an adequate modification of $f(t,\varepsilon)$, so is $\tilde{f}(t,\varepsilon)+c\delta(t)$ with any finite c.

For the particular $1/|t|$ time dependence of the Hamiltonian mentioned in the introduction, one can choose $f(t,\varepsilon)$ as $1/\sqrt{t^2+\varepsilon^2}$, in which case $\tilde{f}(t,\varepsilon)$ becomes

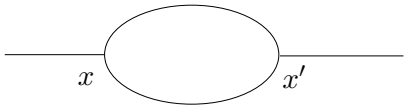
$$f_{1/|t|}(t,\varepsilon) = \frac{1}{\sqrt{t^2+\varepsilon^2}} + 2\ln(\mu\varepsilon)\delta(t). \tag{2.7}$$

It is sometimes more appealing to replace the δ-function in (2.7) by a resolved δ-function, in which case we find

$$f_{1/|t|}(t,\varepsilon) = \frac{1}{\sqrt{t^2+\varepsilon^2}} + 2\ln(\mu\varepsilon)\frac{\varepsilon}{\pi(t^2+\varepsilon^2)} \tag{2.8}$$

(with μ being an arbitrary mass scale).

One should note that it is very natural to think of the above subtraction procedure as renormalizing the singular time dependence of the Hamiltonian. Indeed, the mathematical structure behind generating distributions by means of δ-function subtractions is precisely the same as the one associated with subtracting local counter-terms in order to render conventional field theories finite. For concreteness, consider the one-loop contribution to the full momentum space propagator in $\lambda\phi^3$ field theory, given by the diagram



If we compute it using position space Feynman rules, we find that it is proportional to the Fourier transform of the square of the scalar field Feynman propagator $D(x,x')$. However, while the Feynman propagator itself if a distribution, its square is not. For that reason, if one tries to evaluate the Fourier transform, one obtains infinities, since integrals of $[D(x,x')]^2$ cannot be evaluated. The problem is resolved by subtracting local counter-terms from the field theory Lagrangian, which, for the above diagram, would translate into adding $\delta(x-x')$ and its derivatives (with divergent cutoff-dependent coefficients) to $[D(x,x')]^2$ in such a way as to make it a distribution. The mathematical structure of this procedure is precisely the same as what we employed for renormalizing the singular time dependences in time-dependent Hamiltonians.

We should remark upon the general status of our Hamiltonian prescription viewed against the background of all possible singularity transition recipes one could devise. If the only restriction is that the evolution away from the singularity is given by the original Hamiltonian, one is left with a tremendous infinitefold ambiguity: any unitary transformation can be inserted at $t=0$ and the predictive power is lost completely. One should look for additional principles in order to be able to define a meaningful notion of singularity transition.

Our prescription can be viewed as a very conservative approach, since it preserves the operator structure of the Hamiltonian (the counter-terms added are themselves proportional to h, the singular part of the Hamiltonian). In the absense of further physical specification, this approach appears to be natural and can be viewed as a sort of "minimal subtraction". However, under some circumstances, one may be willing to pursue a broader range of possibilities for defining the singularity transition. For example, one may demand that the resolution of the singular dynamics must have a geometrical interpretation (at finite values of ε). This question will be addressed in [14].

In section 3, the focus of our attention will be a particular quantum system with a Hamiltonian quadratic in the canonical variables. For such linear systems, it is most common to analyze quantum dynamics in the Heisenberg picture, rather than in the Schrödinger

picture we have employed above for the purpose of describing our general formalism. For convenience, we shall give a summary of the relevant derivations in appendix A. In short, one should construct the most general *classical* solution of the system in the form

$$x(t) = Au(t) + A^* u^*(t). \tag{2.9}$$

The solution to the Heisenberg equations of motion is simply obtained by replacing the integration constants A and A^* in the above expression by creation-annihilation operators a and $a^\dagger$, which (with an appropriate normalization of $u(t)$) satisfy the standard commutation relation $[a, a^\dagger] = 1$. The question of solving for the quantum dynamics is then most commonly phrased in terms of constructing the mode functions $u(t)$ and $u^*(t)$, which are normalized solutions to the classical equations of motion.

Our prescription may equally well be applied in such setting. One can analyze the classical equations of motion derived from the time-dependent Hamiltonian. It is safest to do so at finite ε, since the naïve $\varepsilon \to 0$ limit of the classical equations of motion may not necessarily exist. However, the $\varepsilon \to 0$ limit of the solutions for the mode functions *will* exist, and will, of course, define the same quantum dynamics as the general solution to the Schrödinger equation given by (2.2).

3. Free fields on the compactified Milne universe

3.1 The compactified two-dimensional Milne universe

The two-dimensional Milne universe

$$ds^2 = -dt^2 + t^2 dx^2, \tag{3.1}$$

with $0 < t < +\infty$, corresponds to the "future" quadrant $X^\pm > 0$ of Minkowski space $ds^2 = -2dX^+ dX^-$ via the identification

$$X^\pm = \frac{1}{\sqrt{2}} t e^{\pm x}. \tag{3.2}$$

The Milne universe can be compactified by the identification

$$x \sim x + 2\pi, \tag{3.3}$$

which corresponds to the discrete boost identification

$$X^\pm \sim e^{\pm 2\pi} X^\pm. \tag{3.4}$$

The resulting space is a cone, which is singular at its tip $t = 0$.

The action for a free scalar field in the (compactified) Milne universe is

$$S = \int dt\, dx\, t \left(\frac{\dot{\phi}^2}{2} - \frac{\phi'^2}{2t^2} - \frac{m^2 \phi^2}{2} \right). \tag{3.5}$$

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The corresponding equation of motion

$$\ddot{\phi} + \frac{\dot{\phi}}{t} - \frac{\phi''}{t^2} + m^2\phi = 0 \tag{3.6}$$

is solved by

$$\psi_{m,l}(t,x) = \frac{1}{2\sqrt{2}\pi i}\int_{\mathbb{R}} dw\, e^{i\left(\frac{m}{\sqrt{2}}X^- e^{-w} + \frac{m}{\sqrt{2}}X^+ e^{w} + lw\right)} \tag{3.7}$$

$$= \frac{1}{2\sqrt{2}} e^{\frac{l\pi}{2}} e^{-ilx} H^{(1)}_{-il}(mt) \tag{3.8}$$

and their complex conjugates [11, 12]. Here $H^{(1)}$ denotes a Hankel function, and the compactification (3.3) enforces the momentum quantization condition $l \in \mathbb{Z}$. For solutions to the equation of motion (3.6), we define the scalar product [11]

$$(\phi_1, \phi_2) = -i\int_0^{2\pi} dx\, t\, \left[\phi_1(t,x)\dot{\phi}_2^*(t,x) - \dot{\phi}_1(t,x)\phi_2^*(t,x)\right] \tag{3.9}$$

and the Klein-Gordon norm (ϕ, ϕ). The solutions (3.7) are normalized to have Klein-Gordon norm -1.

To quantize the scalar field ϕ, one expands

$$\phi(t,x) = \sum_{k\in\mathbb{Z}} \left[a_k u_k(t,x) + a_k^\dagger u_k^*(t,x)\right], \tag{3.10}$$

where the $u_k(x,t)$ have Klein-Gordon norm 1, which ensures the canonical commutation relations

$$[a_k, a_l^\dagger] = \delta_{k,l}. \tag{3.11}$$

We choose

$$u_k(t,x) = \psi_{m,k}^*(t,x). \tag{3.12}$$

Essentially because $\psi_{m,k}$ of (3.7) are superpositions of negative frequency waves on the covering Minkowski space, the vacuum state defined with the creation and annihilation operators of (3.10) is an adiabatic vacuum of infinite order [11]. Note, however, that in a compactified Milne universe (where globally defined inertial frames are absent) this particular adiabatic vacuum is no more special than any other adiabatic vacuum of infinite order (of which there are infinitely many).

Near $t = 0$, the $l \neq 0$ mode functions behave as (see, for instance, [7])

$$u_l \sim \frac{e^{ilx}}{2\sqrt{2\pi l \sinh(\pi l)}}\left[-\left(\frac{mt}{2}\right)^{il} e^{-\frac{\pi l}{2} - i\varphi_l} + \left(\frac{mt}{2}\right)^{-il} e^{\frac{\pi l}{2} + i\varphi_l}\right], \tag{3.13}$$

with φ_l defined by $e^{i\varphi_l} = \Gamma(1+il)\sqrt{\frac{\sinh(\pi l)}{\pi l}}$ and satisfying $\varphi_{-l} = -\varphi_l$, while

$$u_0 \sim \frac{1}{2\sqrt{2}}\left(1 - \frac{2i}{\pi}\log\left(\frac{mt}{2}\right)\right). \tag{3.14}$$

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The mode functions are clearly singular at $t = 0$. The question we now want to address is whether quantum mechanical evolution can be consistently and naturally defined beyond $t = 0$.

In the literature (see, for instance, [12, 13, 7]), this question has been addressed by extending the range of the t coordinate in the compactified Milne metric (3.1) to $-\infty < t < \infty$, i.e. by adding a "past cone" to the "future cone".[2] In the action (3.5), the factor t is replaced by $|t|$,

$$S = \int dt\, dx\, |t| \left(\frac{\dot{\phi}^2}{2} - \frac{\phi'^2}{2t^2} - \frac{m^2\phi^2}{2} \right). \tag{3.15}$$

The same goes for the scalar product (3.9)

$$(\phi_1, \phi_2) = -i \int_0^{2\pi} dx\, |t| \left[\phi_1(t,x)\dot{\phi}_2^*(t,x) - \dot{\phi}_1(t,x)\phi_2^*(t,x) \right] \tag{3.16}$$

and the corresponding Klein-Gordon norm.

The question then is how to define matching conditions between $t < 0$ mode functions and $t > 0$ mode functions, i.e. how to define global mode functions. Natural globally defined mode functions are obtained by allowing $X^\pm$ to be either both positive or both negative in (3.7) (see (3.12)). As these are superpositions of negative frequency Minkowski modes, they describe excitations above the (adiabatic) vacuum inherited from Minkowski space. The solutions (3.7) have the property that they are analytic in the lower complexified t-plane. For $t < 0$, they can be written as

$$\psi_{m,l}(t,x) = -\frac{1}{2\sqrt{2}} e^{-\frac{l\pi}{2}} e^{-ilx} \left(H_{il}^{(1)}(|mt|) \right)^*, \qquad (t < 0) \tag{3.17}$$

which still has Klein-Gordon norm -1. For t approaching 0 from below, we have for the corresponding mode functions $u_l(t,x) = \psi_{m,l}^*(t,x)$ with $l \neq 0$,

$$u_l \sim \frac{e^{ilx}}{2\sqrt{2\pi l \sinh(\pi l)}} \left(-\left|\frac{mt}{2}\right|^{il} e^{\frac{\pi l}{2} - i\varphi_l} + \left|\frac{mt}{2}\right|^{-il} e^{-\frac{\pi l}{2} + i\varphi_l} \right), \tag{3.18}$$

and

$$u_0 \sim -\frac{1}{2\sqrt{2}} \left(1 + \frac{2i}{\pi} \log\left|\frac{mt}{2}\right| \right). \tag{3.19}$$

Note that, even though the above prescription may seem natural, and it does define consistent matching conditions and a unitary evolution, it should not be given any privileged status. The (compactified) Milne universe contains a genuine singularity at the origin, and the question of how the system evolves in the neighborhood of the singularity cannot be in principle settled through an appeal to a flat Minkowski space (even though there is nothing wrong with using the covering Minkowski space for constructing particular evolutionary prescriptions). As we shall see below, more general rules for singularity

[2] In some string theory contexts, it is natural to consider the full Minkowski space up to the discrete boost identification (3.4), which in addition adds "whisker" regions with closed timelike curves. We will not consider whisker regions in the present paper.

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crossing can be devised, with a different set of mode functions and a different vacuum state (which, being an adiabatic vacuum of infinite order, is no better and no worse than the one inherited from the covering Minkowski space).

Even though the modefunctions $u_l = \psi^*_{m,l}$ constructed above solve the equations of motion derived from the action (3.15) at all positive and all negative t, there are no meaningful equations of motion satisfied at $t = 0$. Correspondingly, even though the quantum evolution defined in terms of the above prescription for the mode functions is unitary (and essentially inherited from the covering Minkowski space), this quantum evolution cannot be represented as a solution to the Schrödinger equation for the Hamiltonian derived from (3.15). In what follows, we shall nevertheless be able to cast this quantum evolution in a Hamiltonian form by appropriately renormalizing the time dependences in the Hamiltonian of the system.

3.2 Quantum Hamiltonian evolution across the Milne singularity

In section 2, we constructed a general prescription which allows to define a Hamiltonian evolution across an isolated singularity in the time dependence of the Hamiltonian. Since the case of a free scalar field on the Milne orbifold falls precisely into this category, it will be instructive to compare the above consideration in terms of the covering Minkowski space with our general prescription. We shall see that the two are in fact related, even though it is only in the parametrization of section 2 that the evolution has a manifestly Hamiltonian form at $t = 0$.

The Hamiltonian corresponding to the action (3.5) is

$$H = \frac{1}{2|t|}\int dx\,\left(\pi_\phi^2 + \phi'^2\right) + \frac{m^2|t|}{2}\int dx\,\phi^2. \tag{3.20}$$

Following the general guidelines presented in section 2, we shall regulate the $1/|t|$ time dependence into $f_{1/|t|}(t,\varepsilon)$ of (2.8):

$$H = \frac{1}{2}f_{1/|t|}(t,\varepsilon)\int dx\,\left(\pi_\phi^2 + \phi'^2\right) + \frac{m^2|t|}{2}\int dx\,\phi^2. \tag{3.21}$$

Near the origin, where the mass term is negligible, the equations of motion take the form

$$\ddot{\phi} - \frac{\dot{f}_{1/|t|}}{f_{1/|t|}}\,\dot{\phi} - f^2_{1/|t|}\phi'' = 0 \tag{3.22}$$

or, after Fourier-expanding $\sqrt{2\pi}\phi(x,t) = \sum \phi_l(t)\exp(ilx)$,

$$\ddot{\phi} - \frac{\dot{f}_{1/|t|}}{f_{1/|t|}}\,\dot{\phi} + l^2 f^2_{1/|t|}\phi = 0. \tag{3.23}$$

The general solution to this equation is

$$\phi_l = A_l \exp\left[il\int f_{1/|t|}(t,\varepsilon)dt\right] + B_l \exp\left[-il\int f_{1/|t|}(t,\varepsilon)dt\right], \tag{3.24}$$

or

$$\phi_l = A_l \exp\left[il\left(\operatorname{arcsinh}\frac{t}{\varepsilon}+\frac{2}{\pi}\ln(\mu\varepsilon)\arctan\frac{t}{\varepsilon}\right)\right]+B_l\left[-il\left(\operatorname{arcsinh}\frac{t}{\varepsilon}+\frac{2}{\pi}\ln(\mu\varepsilon)\arctan\frac{t}{\varepsilon}\right)\right]. \tag{3.25}$$

With ε explicitly taken to 0, this becomes

$$\phi_l = A_l|2\mu t|^{il\,\mathrm{sign}(t)} + B_l|2\mu t|^{-il\,\mathrm{sign}(t)}. \tag{3.26}$$

To construct the Heisenberg field operator (which contains all information on quantum dynamics) one should choose any such complex solution and, after normalizing appropriately, promote it to a mode function, as in (3.10) (see also appendix A).

The question that will interest us here is how the quantum dynamics described by the Hamiltonian with our "minimal subtraction" is related to the mode function prescription (3.12) inherited from the covering Minkowski space. To this end, we shall define mode functions $u_l^{(\mu)}$ that solve (3.23) and coincide with u_l of (3.12) for $t > 0$; however, they will generically differ from u_l for $t < 0$. To see the relation between $u_l^{(\mu)}$ and u_l, we construct $u_l^{(\mu)}$ by choosing A_l and B_l in (3.26) in such a way that it equals (3.13) for $t > 0$ and then compare it, for $t < 0$, with (3.18).

In order to match (3.13) and (3.26) for $t > 0$, we impose

$$A_l = -\frac{1}{2\sqrt{2\pi l\sinh(\pi l)}}\left(\frac{m}{4\mu}\right)^{il} e^{-\frac{\pi l}{2}-i\varphi_l}; \tag{3.27}$$

$$B_l = \frac{1}{2\sqrt{2\pi l\sinh(\pi l)}}\left(\frac{m}{4\mu}\right)^{-il} e^{\frac{\pi l}{2}+i\varphi_l}. \tag{3.28}$$

Then, at $t < 0$,

$$u_l^{(\mu)} = \frac{e^{ilx}}{2\sqrt{2\pi l\sinh(\pi l)}}\left(-\left|\frac{8\mu^2 t}{m}\right|^{-il} e^{-\frac{\pi l}{2}-i\varphi_l} + \left|\frac{8\mu^2 t}{m}\right|^{il} e^{\frac{\pi l}{2}+i\varphi_l}\right). \tag{3.29}$$

Comparing this expression with (3.18), we conclude that they are indeed the same if

$$\mu = \frac{m}{4}\exp\left(\frac{-2\varphi_l+\pi}{2l}\right). \tag{3.30}$$

Note that the fact that μ depends on the Milne momentum l implies that the value of the arbitrary parameter introduced by our renormalization procedure is different for each of the oscillators comprising the field. For that reason, even though the covering Minkowski space prescription turns out to be the same as our "minimal subtraction" for each of the oscillators, for the entire field it is not. Phrased in the Hamiltonian language, the covering space prescription for the Milne singularity transition turns out to be different from the simplest consistent recipe one could devise, even though it is related to such simple recipe in a fairly straightforward way.

4. Conclusions

We have addressed the issue of how one can define a unitary quantum evolution in the presence of isolated singularities in the time dependence of a quantum Hamiltonian. If one demands that the operator structure of the Hamiltonian should be unaffected by regularization prescriptions (the "minimal subtraction" recipe), one discovers a one-parameter family of distinct quantum evolutions across the singularity.

For the case of free quantum fields on the Milne orbifold, the covering Minkowski space considerations previously brought up in the literature [11 – 13, 7] turn out to be closely related to, though distinct from, our "minimal subtraction" proposal. One explicit advantage of our present approach is that it makes the evolution across the singularity manifestly Hamiltonian, which was not the case in the context of the previous discussions.

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A. Linear quantum systems

In this appendix, we shall review the dynamics of linear quantum systems. This material is very basic and well-known; however, it is usually presented in relation to a few specific linear systems of physical interest, whereas, for our purposes, it shall be convenient to summarize here the treatment of a general one-dimensional linear quantum system described by the Hamiltonian

$$H = \frac{f(t)}{2} P^2 + \frac{g(t)}{2} X^2 \tag{A.1}$$

with $f(t)$ and $g(t)$ being arbitrary functions of time.

The equations of motion take the form

$$\dot{P} = -g(t) X \qquad \dot{X} = f(t) P \tag{A.2}$$

or

$$\ddot{X} - \frac{\dot{f}}{f} \dot{X} + f g X = 0. \tag{A.3}$$

Should one succeed finding a complex solution $u(t)$ to this equation, one would be able to write down the most general real solution in the form

$$X(t) = A u(t) + A^* u^*(t) \tag{A.4}$$

with some complex constant A. In the quantum case, the solution to the Heisenberg equations of motion will have the exact same form with A and A^* replaced by Hermitean-conjugate operators a and $a^\dagger$:

$$X_H(t) = a u(t) + a^\dagger u^*(t). \tag{A.5}$$

Our solution for the quantum dynamics shall be complete if we establish the commutation relations for a and $a^\dagger$. Before doing so, we recall the important notion of Wronskian for a linear differential equation. For any two solutions $x_1(t)$ and $x_2(t)$ of a second order differential equation, their Wronskian is defined as

$$W[x_1(t), x_2(t)] = \det \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}. \tag{A.6}$$

It is straightforward to show that, for equation (A.3), the Wronskian of any two given solutions satisfies

$$\dot{W} = \frac{\dot{f}}{f} W. \tag{A.7}$$

In other words, W/f does not depend on time. This circumstance permits to define the "Wronskian norm" for any complex solution $u(t)$:

$$\|u\|_W = -i \frac{W[u, u^*]}{f}. \tag{A.8}$$

As we have just demonstrated, the value of this expression does not depend on the moment of time one chooses to evaluate it. The familiar Klein-Gordon norm for free quantum fields, which we use in section 3, is a direct generalization of the Wronskian norm.

The physical relevance of the Wronskian norm becomes apparent from the consideration of commutators:

$$-i = [P(t), X(t)] = \frac{1}{f}[\dot{X}, X] = \frac{\dot{u}u^* - u\dot{u}^*}{f}[a, a^\dagger] = -i\|u\|_W [a, a^\dagger]. \tag{A.9}$$

Therefore, to obtain the standard commutation relations for the creation-annihilation operators, $[a, a^\dagger] = 1$, one has to choose a complex solution $u(t)$ with Wronskian norm 1.

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